Section 7.4: The Inverse Laplace Transform

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<u>What is the Inverse Laplace Transform?</u>

Discussion

- When we solve DEs…
- Like Antiderivatives
- Non-uniqueness
- Table/Properties

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What is the Inverse Laplace Transform?

Naturally, we prefer to work with continuous functions, since solutions to differential equations are continuous. Fortunately, it can be shown that if two different functions have the same Laplace transform, at most one of them can be continuous.[†] With this in mind we give the following definition.

Inverse Laplace Transform

Definition 4. Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0, \infty)$ and satisfies

(2) $$\mathcal{L}\{f\} = F,$$

then we say that $f(t)$ is the **inverse Laplace transform** of $F(s)$ and employ the notation $f = \mathcal{L}^{-1}\{F\}$.

In case every function $f(t)$ satisfying (2) is discontinuous (and hence not a solution of a differential equation), one could choose any one of them to be the inverse transform; the distinction among them has no physical significance. [Indeed, two *piecewise* continuous functions satisfying (2) can only differ at their points of discontinuity.]

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Example 1 Determine $\mathcal{L}^{-1}\{F\}$, where

(a) $F(s) = \dfrac{2}{s^3}$. **(b)** $F(s) = \dfrac{3}{s^2+9}$. **(c)** $F(s) = \dfrac{s-1}{s^2-2s+5}$.

TABLE 7.1 **Brief Table of Laplace Transforms**

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\dfrac{1}{s}, \quad s > 0$
e^{at}	$\dfrac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\dfrac{b}{s^2+b^2}, \quad s > 0$
$\cos bt$	$\dfrac{s}{s^2+b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$
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Linearity of the Inverse Laplace Transform?

Linearity of the Inverse Transform

Theorem 7. Assume that $\mathcal{L}^{-1}\{F\}$, $\mathcal{L}^{-1}\{F_1\}$, and $\mathcal{L}^{-1}\{F_2\}$ exist and are continuous on $[0, \infty)$ and let c be any constant. Then

(3) $$\mathcal{L}^{-1}\{F_1 + F_2\} = \mathcal{L}^{-1}\{F_1\} + \mathcal{L}^{-1}\{F_2\} ,$$

(4) $$\mathcal{L}^{-1}\{cF\} = c\mathcal{L}^{-1}\{F\} .$$

Proof:

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Example 2 Determine $\mathcal{L}^{-1}\left\{\frac{5}{s-6} - \frac{6s}{s^2+9} + \frac{3}{2s^2+8s+10}\right\}$.

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Example 3 Determine $\mathscr{L}^{-1}\left\{\dfrac{5}{(s+2)^4}\right\}$.

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Example 4 Determine $\mathscr{L}^{-1}\left\{\dfrac{3s+2}{s^2+2s+10}\right\}$.

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Given the choice of finding the inverse Laplace transform of

$$F_1(s) = \frac{7s^2 + 10s - 1}{s^3 + 3s^2 - s - 3}$$

or of

$$F_2(s) = \frac{2}{s-1} + \frac{1}{s+1} + \frac{4}{s+3},$$

which would you select? No doubt $F_2(s)$ is the easier one. Actually, the two functions $F_1(s)$ and $F_2(s)$ are identical. This can be checked by combining the simple fractions that form $F_2(s)$. Thus, if we are faced with the problem of computing $\mathcal{L}^{-1}$ of a rational function such as $F_1(s)$, we will first express it, as we did $F_2(s)$, as a sum of simple rational functions. This is accomplished by the **method of partial fractions.**

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Example 5 Determine $\mathscr{L}^{-1}\{F\}$, where $F(s) = \dfrac{7s-1}{(s+1)(s+2)(s-3)}$.

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Example 6 Determine $\mathcal{L}^{-1}\left\{\dfrac{s^2+9s+2}{(s-1)^2(s+3)}\right\}$.

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Example 7 Determine $\mathscr{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$.

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